

Semigroup cohomology and applications

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This article is a survey of the author's research. It consists of three sections concerned three kinds of cohomologies of semigroups. Section 1 considers 'classic' cohomology as it was introduced by Eilenberg and MacLane. Here the attention is concentrated mainly on semigroups having cohomological dimension 1. In Section 2 a generalization of the Eilenberg–MacLane cohomology is introduced, the so-called 0-cohomology, which appears in applied topics (projective representations of semigroups, Brauer monoids). At last Section 3 is devoted to further generalizing: partial cohomology defined and discussed in it are used then for calculation of the classic cohomology for some semigroups.

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1 EM-cohomology

In this section we deal with the Eilenberg–MacLane cohomology of semigroups [3] (shortly EM-cohomology). Its definition is the same as for groups:

$$H^n(S, A) = \text{Ext}_{\mathbf{Z}S}^n(\mathbf{Z}, A).$$

Here S is a semigroup, A a left S -module (i. e. a left module over the integral semigroup ring $\mathbf{Z}S$), \mathbf{Z} is considered as a trivial S -module (i. e. $xa = a$ for all $x \in S$, $a \in \mathbf{Z}$); $H^n(S, A)$ is called a n^{th} cohomology group of S (with the coefficient module A).

Another definition (equivalent to preceding one) of $H^n(S, A)$ is following. Denote by $C^n(S, A)$ the group of all maps $f : \underbrace{S \times \dots \times S}_{n \text{ times}} \rightarrow A$ (a group of

n -cochains); a coboundary homomorphism $\partial^n : C^n(S, A) \rightarrow C^{n+1}(S, A)$ is given by the formula

$$\begin{aligned} \partial^n f(x_1, \dots, x_{n+1}) &= x_1 f(x_2, \dots, x_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i f(x_1, \dots, x_i x_{i+1}, \dots, x_{n+1}) + (-1)^{n+1} f(x_1, \dots, x_n) \end{aligned} \quad (1)$$

Then $\partial^n \partial^{n-1} = 0$, i. e.

$$\text{Im} \partial^{n-1} = B^n(S, A) \text{ (a group of coboundaries)}$$

$$\subseteq \text{Ker} \partial^n = Z^n(S, A) \text{ (a group of cocycles)}$$

and cohomology groups are defined as $H^n(S, A) = Z^n(S, A) / B^n(S, A)$.

It is worth noting two simple properties of the semigroup cohomology [1] which are useful below.

(1) Adjoin to a given semigroup S an extra element 1 and extend the operation (multiplication) of S to $S^1 = S \cup \{1\}$ by

$$\forall s \in S \quad s \cdot 1 = 1 \cdot s = s, \quad 1 \cdot 1 = 1.$$

Then S^1 becomes a monoid (i. e. a semigroup with an identity element) and is called *a semigroup with an adjoint identity*. Every S -module turns naturally into a (unitary) S^1 -module and we have

$$H^n(S^1, A) \cong H^n(S, A), \quad n \geq 0.$$

(2) If S possesses a zero element 0 then $H^n(S, A) = 0$ for each S -module A and for all $n \geq 1$. In particular, for any semigroup S we can define *the semigroup S^0 with an adjoint zero* (analogously to S^1); then we have $H^n(S^0, A) = 0$ ($n \geq 1$).

The semigroup EM-cohomology has not so wide applications as the cohomology of groups. Nevertheless it is interesting for homologists at least as a model for testing homological methods. The problem of describing semigroups having cohomological dimension 1 is such an example. This problem has its own story.

Cohomological dimension (c. d.) of a semigroup S is the maximal integer n such that $H^n(S, A) \neq 0$ for some S -module A . There are many reasons to study algebraic objects of c. d. 1 (see, e. g., [4]).

It is an easy exercise to prove that both a free group and a free semigroup (or a free monoid) have c.d. 1 [3]. For groups the converse is true — this is the well-known Stallings–Swan theorem [2]. So a group has cohomological dimension one if and only if it is free.

Now, what about semigroups?

First, as we have mentioned above, every semigroup with 0 has c.d. ≤ 1 . This is the main reason why we have to confine ourselves to considering cancellative semigroups only.

Second, a free group is not free as a semigroup. So even for cancellative semigroups the Stallings–Swan theorem doesn't hold.

B. Mitchell [13] has shown that a so-called partially free monoid (the free product of a free group and a free monoid) has c.d. 1. He has supposed that if c.d. $S = 1$ for S cancellative then S is partially free.

In [16] I have built the first counter-example for Mitchell conjecture:

$$S = \langle a, b, c, d \mid ab = cd \rangle \quad (2)$$

and in [17] I have formulated a ‘weakened Mitchell conjecture’. It turned out true:

Theorem 1 [22] *Every cancellative semigroup of c.d. 1 can be embedded into a free group.*

In the proof of this theorem the passage from the homological language to the semigroup one is realized by the following lemma (which may be helpful not only for semigroups).

Let A be a left module over an arbitrary ring R ,

$$\dots \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \longrightarrow 0$$

its projective resolution. Evidently, d_n may be considered as a $(n + 1)$ -dimensional cocycle with values in the R -module $\text{Im } d_n$.

Lemma 2 *The cocycle $d_n \in Z^{n+1}(A, \text{Im } d_n)$ is a coboundary iff the projective dimension of A is not greater than n .*

Applying Lemma 2 to a bar-resolution of the S -module \mathbf{Z} we obtain the next property of a cancellative semigroup S with c.d. 1. Consider a graph

with the elements of S as vertices and with the pairs $(a, b) \in S \times S$ such that $aS \cap bS \neq \emptyset$ as edges. Then every circuit of this graph is triangulable.

This property allows to prove that S can be embedded into a group (the latter turns out free by the Stallings–Swan theorem). Note that the converse assertion is certainly not true: there is a subsemigroup of a free semigroup, which has c.d. 1, while c.d. of its anti-isomorphic is equal to 2 (see the examples below). A nice answer is only obtained in the commutative case [19]: the c.d. of a commutative cancellative semigroup is equal to 1 if and only if this semigroup can be embedded into \mathbf{Z} .

So a new problem arises: to describe subsemigroups of a free group having c.d. 1. This question seems rather difficult even if we restrict ourselves to subsemigroups of a free semigroup. The following results are taken out of [24].

Let S be a subsemigroup of a free semigroup F . Further development of the proof of Theorem 1 gives us the next assertion:

Theorem 3 *Let c.d. $S = 1$ and $aS \cap S \neq \emptyset \neq Sa \cap S$ for some $a \in F \setminus S$. There exists such $x \in aS \cap S$ that $aS \cap S \subset xF^1$.*

This theorem allows to build a lot of subsemigroups of F having c.d. > 1 .

Example 1 Let a, p, q, r are different elements of F such that:

- 1) $\min(|a|, |p|, |q|, |r|) = |a|$ ($|a|$ denotes the length of the word a),
- 2) p and q begin with different letters.

Then the subsemigroup $S = \langle p, q, r, ap, aq, ra \rangle$ has c.d. > 1 .

From Theorem 3 a solution of the proposed problem for left ideals is obtained:

Proposition 4 *A left ideal of a free semigroup has c.d. 1 iff it is free.*

Corollary 5 *Every proper two-sided ideal of a free semigroup has c.d. > 1 .*

For principal right ideals the situation is similar:

Proposition 6 *A principal right ideal of a free semigroup has c.d. 1 iff it is free.*

However for arbitrary right ideals the analog of Proposition 4 is not true:

Example 2 Let $F = \langle a, b \rangle$ be a free semigroup. Then $R = \{b, aba\}F^1$ is not free but $\text{c.d.} R = 1$.

By the way, from these results a counter-example to another conjecture follows. Yu. Drozd supposed that for any $S \in F$ either S or the antiisomorphic to S has $\text{c.d.} 1$. Consider the principal left ideal $L = F^1 aba$ in a free semigroup $F = \langle a, b \rangle$. It is not free since its generators $aba, (ab)^2, (ab)^2 a, (ab)^3$ obey the relation

$$(ab)^2 \cdot (ab)^2 a = (ab)^3 \cdot aba.$$

By Proposition 4 $\text{c.d.} L > 1$. Of course its antiisomorphic $R = abaF^1$ is not free too and $\text{c.d.} R > 1$ by Proposition 6. Hence the pair (L, R) gives a counter-example to the conjecture.

2 0-cohomology

In order to see how 0-cohomology appears let us try to define a projective representation of a semigroup.

Let K be a field, K^\times its multiplicative group, n a positive integer, $M(n, K)$ the semigroup of all $n \times n$ matrices over K . Define an equivalence: for $A, B \in M(n, K)$

$$A \sim B \iff \exists \lambda \in K^\times \ A = \lambda B.$$

Then \sim is a congruence on the semigroup $M(n, K)$ and we can consider a factor semigroup $PM(n, K) = M(n, K)/\sim$, ‘the projective linear semigroup’.

Like for groups we call a *projective representation* of a given semigroup S a homomorphism $\Gamma : S \rightarrow PM(n, K)$.

Fix an element in each \sim -class. Then Γ induces a map $\Gamma' : S \rightarrow M(n, K)$. Now we can redefine a projective representation of S : it is a map $\Gamma' : S \rightarrow M(n, K)$ such that

- 1) $\Gamma'(x)\Gamma'(y) = 0 \iff \Gamma'(xy) = 0$,
- 2) $\Gamma'(x)\Gamma'(y) = \Gamma'(xy)\rho(x, y)$, where $\rho : S \times S \rightarrow K^\times$ is a partial function defined on the subset $\{(x, y) \mid \Gamma'(xy) \neq 0\}$.

Certainly ρ yields the equation

$$\rho(x, y)\rho(xy, z) = \rho(x, yz)\rho(y, z)$$

for $\Gamma'(xyz) \neq 0$ and can be used as the corresponding 2-cocycle (like a factor system in Group Theory) excepting its partiality. Therefore we must anew define suitable cohomology as follows.

Let S be an arbitrary semigroup with a zero. An Abelian group A is called a *0-module* over S , if an action $(S \setminus \{0\}) \times A \rightarrow A$ is defined which satisfies for all $s, t \in S \setminus \{0\}$, $a, b \in A$ the following conditions:

$$s(a + b) = sa + sb,$$

$$st \neq 0 \implies s(ta) = (st)a.$$

A *n-dimensional 0-cochain* is a partial n -place map from S to A which is defined for all n -tuples (s_1, \dots, s_n) , such that $s_1 \cdot \dots \cdot s_n \neq 0$. A coboundary homomorphism is given like for the usual cohomology by the formula (1). The equality $\partial^n \partial^{n-1} = 0$ is valid too. We denote

$$\text{Im} \partial^{n-1} = B_0^n(S, A) \text{ (a group of 0-coboundaries)}$$

$$\subseteq \text{Ker} \partial^n = Z_0^n(S, A) \text{ (a group of 0-cocycles)}$$

and *0-cohomology groups* are defined as $H_0^n(S, A) = Z_0^n(S, A) / B_0^n(S, A)$.

Note that for a semigroup $T^0 = T \cup \{0\}$ with an adjoined zero

$$H_0^n(T^0, A) \cong H^n(T, A),$$

so the 0-cohomology may be considered as a generalization of the Eilenberg–MacLane cohomology.

Properties of 0-cohomology are not considered here since they follow from the properties of partial cohomologies (see Section 3).

Before returning to the projective representations we need a semigroup-theoretic construction, the so-called semilattice of groups [6].

Let Λ be a semilattice (i. e. a partially ordered set in which every two elements λ, μ have the greatest lower bound $\lambda\mu$) and let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of disjoint groups. For each pair $\lambda, \mu \in \Lambda$ such that $\lambda \geq \mu$, let $\varphi_\mu^\lambda : G_\lambda \rightarrow G_\mu$ be a homomorphism. Suppose that

1) φ_λ^λ is identical for every $\lambda \in \Lambda$,

2) $\varphi_\mu^\lambda \varphi_\nu^\mu = \varphi_\nu^\lambda$ for all $\lambda \geq \mu \geq \nu$.

Define a multiplication on the set $T = \bigcup_{\lambda \in \Lambda} G_\lambda$ by the rule: if $x \in G_\lambda$, $y \in G_\mu$

$$xy = (\varphi_{\lambda\mu}^\lambda x)(\varphi_{\lambda\mu}^\mu y).$$

Then T becomes a semigroup which is called a *semilattice of groups*.

Now return to the projective representations. Recall [8] that if S is a group then one defines an equivalence on the set of the factor systems of S (which corresponds to the equivalence of projective representations); the factor set by this equivalence is a group $\text{Sch}(S, K)$ which is called a *Schur multiplier* and describes (in some sense) all projective representations of S over K . It is well-known that $\text{Sch}(S, K) \cong H^2(S, K^\times)$, where K^\times is considered as a trivial S -module.

What will be for semigroups? In this situation $\text{Sch}(S, K)$ is not a group (more exactly, it becomes an inverse semigroup). Let Λ be a semilattice of all two-sided ideals of S (including \emptyset and S) with respect to the inclusion and the union as a greatest lower bound. Then restriction of 0-cochains induces homomorphisms

$$\varphi_J^I : H_0^n(S/I, K^\times) \longrightarrow H_0^n(S/J, K^\times)$$

for ideals $I \subseteq J$ and we have a semilattice of groups $\bigcup_{I \in \Lambda} H_0^n(S/I, K^\times)$ (here for $I = \emptyset$ we set $H_0^n(S/\emptyset, K^\times) = H^n(S, K^\times)$). The next assertion was proved in [15]:

Theorem 7 *For every semigroup S and every field K*

$$\text{Sch}(S, K) \cong \bigcup_{I \in \Lambda} H_0^2(S/I, K^\times)$$

Note that even if $0 \notin S$ we have to use 0-cohomology for describing of $\text{Sch}(S, K)$.

Another application of the 0-cohomology appears in connection with the Brauer monoid. This notion was introduced by Haile, Larson and Sweedler [9], [10] while they studied the so-called strongly primary algebras (a generalization of central simple ones). I shall not give their original definition which is rather complicated. But it turned out that the Brauer monoid can

be defined in terms of the 0-cohomology [20]. To do it one must introduce a new notion, a modification of a group.

By a *modification* $G(*)$ of a group G we mean a semigroup on the set $G^0 = G \cup \{0\}$ with an operation $*$ such that $x * y$ is equal either to xy or to 0, while

$$0 * x = x * 0 = 0 * 0 = 0$$

and the identity of G is the same for the semigroup $G(*)$.

In other words, to obtain a modification, one must erase the contents of some inputs in the multiplication table of G and insert there zeros so that the new operation would be associative.

Note some general properties of modifications. First, a modification of G satisfies the weak cancellation condition: from $x * z = y * z \neq 0$ it follows $x = y$ and analogously for left cancellation. Second, let U be a subgroup of all invertible elements in $G(*)$. Then its complement $I = G(*) \setminus U$ is a two-sided ideal. One can show that if G is finite, I is nilpotent.

Let $S = G(*)$ and $T = G(\star)$ be modifications of G . It is clear that $S \cap T = G(\circ)$ is a modification too, where

$$x \circ y \neq 0 \iff x * y \neq 0 \neq x \star y.$$

We write $S \prec T$ if $x \star y = 0$ implies $x * y = 0$ for all $x, y \in G$. Obviously, all modifications of G constitute a semilattice $M(G)$: the greatest lower bound in it is $S \cap T$.

Each G -module A can be turned into a 0-module over a modification S in a natural way. Moreover, if $S \prec T$ then each 0-module over T is transformed into a 0-module over S . Therefore for $S \prec T$ a homomorphism is defined

$$\varphi_S^T : H_0^n(T, A) \longrightarrow H_0^n(S, A)$$

and we obtain a semilattice of groups $\bigcup_{S \in M(G)} H_0^n(S, A)$.

In particular, let L be a finite-dimensional normal extension of a field K with the Galois group G . Then L^\times is a G -module. We define a *(relative) Brauer monoid* as

$$\text{Br}(G, L) = \bigcup_{S \in M(G)} H_0^2(S, L^\times)$$

(the adjective ‘relative’ will be omitted since in this article relative Brauer monoids are only considered).

In the case when operation $*$ is defined in such a way that $x * y = xy$ for $x, y \neq 0$, we have

$$H_0^2(S, L^\times) \cong H^2(G, L^\times),$$

so the Brauer group is a subgroup of the Brauer monoid.

One can hope that the Brauer monoid will be useful. For example, it is well-known that the Brauer group is trivial for any finite field whereas the Brauer monoid is not trivial for each non-trivial field extension.

The Brauer monoid classifies strongly primary algebras over a field like the Brauer group classifies division algebras.

The use of 0-cohomology allows us to split the study of the Brauer monoid into two problems:

- 1) describing all modifications of a given finite group,
- 2) computing 0-cohomology of a modification.

Both of them seem rather difficult, especially the first. Its solution is unknown even for cyclic groups. In [21] some class of modifications of simple cyclic groups is described. It implies that the number of modifications of the group \mathbf{Z}_p is $O(p^2)$. All multiplication tables of the modifications S_1, \dots, S_{15} of \mathbf{Z}_5 (up to automorphisms of the group) are shown in Table 1; for \mathbf{Z}_7 their number equals 145.

As to the second problem, the initial step in solving it may consist in eliminating the influence of invertible elements of modifications on the structure of the Brauer monoid. Some results in this direction were obtained in [12], [20].

As above let G be the Galois group of a finite-dimensional extension L/K , $S = G(*)$ its modification, U the subgroup of invertible elements of S , P the subfield of all U -fixed elements: $P = \{a \in L \mid Ua = a\}$. The inclusion $U \hookrightarrow S$ induces a homomorphism

$$\psi : H_0^2(S, L^\times) \longrightarrow H^2(U, L^\times)$$

We shall study this homomorphism in the situation when U is a normal subgroup of S (i. e. $x * U = U * x$ for all $x \in S$). Then the factor semigroup S/U is well-defined.

Further, if $U \triangleleft S$ then P^\times is a 0-module over S/U . The inclusion $P^\times \hookrightarrow L^\times$ and the epimorphism $S \rightarrow S/U$ induce a homomorphism

$$\chi : H_0^2(S/U, P^\times) \longrightarrow H_0^2(S, L^\times)$$

S_1	a^1	a^2	a^3	a^4
a^1	0	0	0	0
a^2	0	a^4	0	a^1
a^3	0	0	a^1	0
a^4	0	a^1	0	0

S_2	a^1	a^2	a^3	a^4
a^1	0	0	0	0
a^2	0	a^4	0	0
a^3	0	0	a^1	0
a^4	0	0	0	0

S_3	a^1	a^2	a^3	a^4
a^1	0	0	0	0
a^2	0	0	0	a^1
a^3	0	0	a^1	a^2
a^4	0	a^1	a^2	a^3

S_4	a^1	a^2	a^3	a^4
a^1	0	0	0	0
a^2	0	0	0	a^1
a^3	0	0	a^1	0
a^4	0	a^1	0	0
S_5	a^1	a^2	a^3	a^4
a^1	0	0	0	0
a^2	0	0	0	a^1
a^3	0	0	a^1	0
a^4	0	0	0	0
S_6	a^1	a^2	a^3	a^4
a^1	0	0	0	0
a^2	0	0	0	0
a^3	0	0	a^1	a^2
a^4	0	0	a^2	0
S_7	a^1	a^2	a^3	a^4
a^1	0	0	0	0
a^2	0	0	0	0
a^3	0	0	a^1	0
a^4	0	a^1	0	0
S_8	a^1	a^2	a^3	a^4
a^1	0	0	0	0
a^2	0	0	0	0
a^3	0	0	a^1	0
a^4	0	0	a^2	0
S_9	a^1	a^2	a^3	a^4
a^1	0	0	0	0
a^2	0	0	0	0
a^3	0	0	0	a^2
a^4	0	0	a^2	0
S_{10}	a^1	a^2	a^3	a^4
a^1	0	0	0	0
a^2	0	0	0	0
a^3	0	0	0	a^2
a^4	0	0	a^2	a^3
S_{11}	a^1	a^2	a^3	a^4
a^1	0	0	0	0
a^2	0	0	0	0
a^3	0	0	0	0
a^4	0	a^1	0	0
S_{12}	a^1	a^2	a^3	a^4
a^1	0	0	0	0
a^2	0	0	0	0
a^3	0	0	0	0
a^4	0	a^1	0	a^3
S_{13}	a^1	a^2	a^3	a^4
a^1	0	0	0	0
a^2	0	0	0	0
a^3	0	0	0	0
a^4	0	0	a^2	0
S_{14}	a^1	a^2	a^3	a^4
a^1	0	0	0	0
a^2	0	0	0	0
a^3	0	0	0	0
a^4	0	0	0	0
S_{15}	a^1	a^2	a^3	a^4
a^1	0	0	0	0
a^2	0	0	0	0
a^3	0	0	0	0
a^4	0	0	0	a^3

Figure 1: The modifications of \mathbf{Z}_5

Theorem 8 *Let $U \triangleleft S$. Then the sequence*

$$0 \longrightarrow H_0^2(S/U, P^\times) \xrightarrow{\chi} H_0^2(S, L^\times) \xrightarrow{\psi} H^2(U, L^\times)$$

is exact.

Corollary 9 *If the field L is finite then*

$$H_0^2(S, L^\times) \cong H_0^2(S/U, P^\times)$$

Therefore, for finite fields the problem is reduced to the computation of 0-cohomology of a nilpotent 0-cancellative semigroup $(S/U) \setminus \{1\}$. I believe that such an algorithm can be built.

At last note that W. Clark [5] used 0-cohomology (however with trivial 0-modules only) for investigation of some matrix algebras.

3 Partial cohomologies

0-Cohomology has one more application: for calculating of EM-cohomology. However from this point of view it is worth once more to generalize our construction.

One can ask: what would be if we considered partial maps as cochains, starting from an arbitrary subset $W \subseteq S$, not necessary from $S \setminus \{0\}$? It was shown in [17] that this question is reduced to the following particular case.

Let a semigroup S be generated by a subset W with defining relations of the form $xy = z$ for some $x, y, z \in W$. Such a W will be called a *root* of S . We denote by W_n a set of all n -tuples (x_1, \dots, x_n) such that $x_i x_{i+1} \dots x_j \in W$ for all $1 \leq i \leq j \leq n$. Every map from W_n to a S -module A is called a *partial n -dimensional cochain* of W or a *W -cochain* with values in A . n -Dimensional W -cochains form an Abelian group $C^n(S, W, A)$ for $n > 0$. We set $C^0(S, W, A) = A$, and if $W_n = \emptyset$ then $C^n(S, W, A) = 0$. The coboundary homomorphism is given by the same formula (1); the corresponding *partial cohomology groups* (or *W -cohomology groups*) are denoted by $H^n(S, W, A)$.

It is clear that we obtain EM-cohomology if $W = S$. Reducing 0-cohomology to a partial one looks more complicated: if S is a semigroup with 0, $W = S \setminus \{0\}$, then we generate a new semigroup $T = \langle W \rangle$ with the

operation $*$ and defining relations of the form $u * v = w$, where $u, v, w \in W$ and $uv = w$ in S . Then W is a root in T and

$$H_0^n(S, A) \cong H^n(T, W, A).$$

Having a presentation of a semigroup S one can easily build some of its roots.

Example 3 Let $S = \langle a, b, c, d \mid ab = cd \rangle$ (see (2)). Then

$$W = \{a, b, c, d, x = ab\}, \quad W_2 = \{(a, b), (c, d)\}, \quad W_3 = \emptyset$$

and

$$S = \langle a, b, c, d, x \mid ab = x, cd = x \rangle$$

How are $H^n(S, W, A)$ and $H^n(S, A)$ connected?

The embedding $W \hookrightarrow S$ induces homomorphisms

$$\theta_W^n : H^n(S, A) \longrightarrow H^n(S, W, A),$$

Proposition 10 [18] *If W is a root of a semigroup S then θ_W^n is an isomorphism for $n \leq 1$ and a monomorphism for $n = 2$.*

Generally speaking, θ_W^2 can be non-surjective (by the way it means that partial cohomology ought not be a derived functor in the category of S -modules).

Proposition 10 enables us to use partial cohomology for calculating 1-dimensional EM-cohomology of semigroups (and getting some information about 2-dimensional one) in the case when one succeeds to find a ‘good’ root in a given semigroup. For instance, consider the semigroup S from Example 3. Define for each $f \in Z^2(S, W, A)$ the 1-dimensional W -cochain h by

$$h(s) = \begin{cases} f(a, b), & \text{if } s = a, \\ f(c, d), & \text{if } s = c, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

Then $f = \partial h$, so $H^2(S, W, A) = 0$. Since θ_W^2 is injective, $H^2(S, A) = 0$ too.

To study θ^n for $n > 1$ we need some new definitions.

Let S be a semigroup, W be its root. A decomposition $x = x_1 \dots x_k$ ($x_i \in W$) of an element $x \in S \setminus W$ is called *reduced* if $x_i x_{i+1} \dots x_j \notin W$

for each i, j , $1 \leq i < j \leq k$. We mean that a reduced decomposition of an element $x \in W$ is its decomposition into the product of one multiplier. A root W is said to be *canonic* if each element $x \in S$ has the unique reduced decomposition.

For example, the set of all element of S is a canonic root.

A root W is called a *J-root* if $xy = x$, $yz = z$ implies $xz \in W$ for all $x, y, z \in W$.

Theorem 11 [18, 23] *If W is a canonic J-root of S , then θ_W^n are isomorphisms for all $n \geq 0$.*

As above, we can use Theorem 11 for calculating EM-cohomology in higher dimensions. For example, if $S = T * U$ is the free product of semigroups T and U , then $W = T \cup U$ is its canonic *J-root* and $W_n = T_n \cup U_n$. Thus, we get

$$H^n(S, A) \cong H^n(S, W, A) \cong H^n(T, A) \oplus H^n(U, A)$$

for every S -module A . Below we consider less trivial examples.

Example 4 Let $S = \langle a, b_1, b_2, \dots \mid aP = Q \rangle$ be such a semigroup that the words P and Q do not contain the letter a . Denote $W = F \cup \{a\}$, where $F = \langle b_1, b_2, \dots \rangle$ is a subsemigroup of S . Then W is a canonic *J-root*. The fact that F is free facilitates the calculation of W -cohomology of S ; so we obtain $H^2(S, A) = 0$ for every X -module A (that is c.d. $S = 1$).

Example 5 The semigroup

$$S^{\text{op}} = \langle a, b_1, b_2, \dots \mid Pa = Q \rangle,$$

is antiisomorphic to S (see Example 4). Like for S , the subset $W = F \cup \{a\}$ is a canonic *J-root*. However c.d. $S^{\text{op}} = 2$. Besides, $H^2(S^{\text{op}}, A) \cong A/B$, where

$$B = PA + \sum_i \left(\frac{\partial P}{\partial b_i} - \frac{\partial Q}{\partial b_i} \right) A;$$

here $\frac{\partial}{\partial b}$ is an analog of the Fox' derivative [7] adapted to semigroups in [18].

Consider one more pair of antiisomorphic semigroups.

Example 6 Let U be an arbitrary semigroup,

$$T = \langle U, p \mid Up = p \rangle \quad (p \notin U)$$

This notation means that T is generated by its subsemigroup U and by an element $p \notin U$ and is defined by relations of the form

$$u \cdot v = uv, \quad u \cdot p = p \quad (u, v \in U)$$

The subset $W = U \cup \{p\}$ turns out a canonic J -root. We get c.d. $T = 1$ and $H^1(T, A) \cong A/(p-1)A$ for every T -module A .

Example 7 Now consider the semigroup

$$T^{\text{op}} = \langle U, p \mid pU = p \rangle,$$

antiisomorphic to T . Its EM-cohomology is much more complicated:

Proposition 12 *Let A be a T^{op} -module, A_1 be its additive group considered as a trivial T^{op} -module. The homomorphisms $\psi^n : H^n(T^{\text{op}}, A) \rightarrow H^n(U, A)$ induced by the embedding $U \hookrightarrow T^{\text{op}}$ are inserted into the long exact sequence*

$$\begin{aligned} 0 \rightarrow H^0(T^{\text{op}}, A) \xrightarrow{\psi^0} H^0(U, A) \rightarrow H^0(U, A_1) \rightarrow H^1(T^{\text{op}}, A) \xrightarrow{\psi^1} \dots \\ \dots \rightarrow H^n(T^{\text{op}}, A) \xrightarrow{\psi^n} H^n(U, A) \rightarrow H^n(U, A_1) \rightarrow \dots \end{aligned} \quad (4)$$

By the last two examples we can build a semigroup T such that c.d. $T = 1$ and c.d. $T^{\text{op}} = \infty$. To do it take the additive group of the ring \mathbf{Z}_9 as A and its multiplicative group as U . The action of U on A coincides with the multiplication in \mathbf{Z}_9 . Then $H^n(T^{\text{op}}, A) \cong \mathbf{Z}_3$ for $n > 1$.

It is worth to add that the notion of a canonic root can be applied to algorithmic problems. For instance, with its help a new family of semigroups with solvable word problem was obtained [11].

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